

Day 3: Greatest Common Divisor (GCD) and Least Common Multiple (LCM)

Aleksandre Katsia

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Theory

VI. Greatest Common Divisor (GCD). A common divisor of several natural numbers is a number that is a divisor of each of them.

The *greatest common divisor* of natural numbers $k, m, \dots, n$ is the largest number among their common divisors, and it is denoted by $D(k, m, \dots, n)$ or $\gcd(k, m, \dots, n)$.

If $D(m, n) = 1$, then m and n are called *coprime* (or relatively prime).

Note that two prime numbers are always coprime, but two composite numbers can also be coprime, such as 8 and 9.

Example. Find $D(126, 540, 630)$.

Solution. Decompose the given numbers into prime factors:

$$\begin{array}{r|l}
 126 & 2 \\
 63 & 3 \\
 21 & 3 \\
 7 & 7 \\
 1 & \\
 \hline
 \end{array}
 \qquad
 \begin{array}{r|l}
 540 & 2 \\
 270 & 2 \\
 135 & 3 \\
 45 & 3 \\
 15 & 3 \\
 5 & 5 \\
 1 & \\
 \hline
 \end{array}
 \qquad
 \begin{array}{r|l}
 630 & 2 \\
 315 & 3 \\
 105 & 3 \\
 35 & 5 \\
 7 & 7 \\
 1 & \\
 \hline
 \end{array}$$

That is,

$$126 = 2 \cdot 3^2 \cdot 7, \quad 540 = 2^2 \cdot 3^3 \cdot 5, \quad 630 = 2 \cdot 3^2 \cdot 5 \cdot 7.$$

In these prime factorizations, 2 appears as a common factor once, 3 appears twice, while 5 and 7 are not common to all three numbers. Therefore:

$$D(126, 540, 630) = 2 \cdot 3 \cdot 3 = 18.$$

VII. Least Common Multiple (LCM). A common multiple of several natural numbers is a number that is a multiple of each of them.

The *least common multiple* of natural numbers $k, m, \dots, n$ is the smallest number among their positive common multiples, and it is denoted by $K(k, m, \dots, n)$ or $\text{lcm}(k, m, \dots, n)$.

Example. Find $K(270, 300, 315)$.

Solution. Decompose the given numbers into prime factors:

$$\begin{array}{r|l}
 270 & 2 \\
 135 & 3 \\
 45 & 3 \\
 15 & 3 \\
 5 & 5 \\
 1 &
 \end{array}
 \qquad
 \begin{array}{r|l}
 300 & 2 \\
 150 & 2 \\
 75 & 3 \\
 25 & 5 \\
 5 & 5 \\
 1 &
 \end{array}
 \qquad
 \begin{array}{r|l}
 315 & 3 \\
 105 & 3 \\
 35 & 5 \\
 7 & 7 \\
 1 &
 \end{array}$$

That is,

$$270 = 2 \cdot 3^3 \cdot 5, \quad 300 = 2^2 \cdot 3 \cdot 5^2, \quad 315 = 3^2 \cdot 5 \cdot 7.$$

Therefore:

$$K(270, 300, 315) = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7 = 18900.$$

VIII. Relationship Between GCD and LCM. For any two positive integers a and b , the product of their greatest common divisor and least common multiple is equal to the product of the numbers themselves:

$$\gcd(a, b) \cdot \text{lcm}(a, b) = a \cdot b.$$

III. The Euclidean Algorithm (Subtraction Method). The Euclidean algorithm is an efficient method for finding the greatest common divisor (gcd) of two natural numbers. Instead of using prime factorization, we repeatedly subtract the smaller number from the larger number until both numbers become equal. That common number is the gcd.

$$\gcd(a, b) = \gcd(a - b, b) \quad \text{where } a > b$$

Example. Find $\gcd(18, 48)$:

$$\gcd(18, 48) = \gcd(18, 30) = \gcd(18, 12) = \gcd(6, 12) = \gcd(6, 6) = 6$$

Textbook Problems

1. **(1.13)** Find the greatest common divisor using prime factorizations:

- 1) 12 and 42;
- 2) 240 and 360;
- 3) 1680 and 4200;
- 4) 180, 270, and 450.

2. **(1.14)** Find the least common multiple using prime factorizations:

- 1) 12 and 15;

2) 24, 30, and 60;

3) 7, 8, and 13;

4) 20, 65, and 125.

3. (1.15) Find the greatest common divisor:

1) $2^5 \cdot 3^7 \cdot 5$ and $3^3 \cdot 5^2$;

2) $2 \cdot 7^2 \cdot 3^2$ and $2^2 \cdot 7^3$.

4. (1.7) Find the greatest common divisor:

34 and 48; 120 and 150; 450 and 540; 14, 20, and 48; 36, 54, and 72; 88, 240, and 300.

5. (1.8) Find the least common multiple:

8 and 28; 64 and 96; 240 and 360; 480 and 540; 48, 72, and 80; 150, 300, and 600.

Word Problems & Applications

W1. At the gym, Hillary swims every 6 days, runs every 4 days, and cycles every 16 days. If she did all three activities today, in how many days will she do all three activities again on the same day?

W2. Oscar needs to ship 14 rock CDs, 12 classical CDs, and 8 pop CDs. He can pack only one type of CD in each box, and he must pack the same number of CDs in each box. What is the greatest number of CDs Oscar can pack in each box?

W3. I want to plant 45 sunflower plants, 81 corn plants, and 63 tomato plants in my garden. If I put the same number of plants in each row and each row has only one type of plant, what is the greatest number of plants I can put in one row?

Additional Practice Problems

A1. Prove that the GCD of two consecutive integers is always 1.

A2. Find the GCD of 1001 and 143 using the Euclidean algorithm.

A3. The GCD of two numbers is 36 and their LCM is 720. If one of the numbers is 180, what is the other number?

A4. The GCD of two numbers is 20 and their LCM is 300. If one of the numbers is 60, what is the other number?

- A5.** A farmer has two pieces of wire: one measuring 150 meters and the other 225 meters. What is the largest length of wire that can be cut from both pieces without any leftover?
- A6.** Use the Euclidean algorithm to find $\gcd(12345, 11111)$.
- A7.** Prove that if a and b are natural numbers, then $\gcd(a, a+2)$ is either 1 or 2. Under what conditions is it equal to 2?
- A8.** Two non-zero natural numbers x and y satisfy $\gcd(x, y) = 12$ and $\text{lcm}(x, y) = 180$. Find all possible pairs (x, y) such that $x \leq y$.
- A9.** What can be the gcd of two integers with difference 3? 5? 9? 18?
- A10.** The difference between two integers is 7. What is the gcd if the integers if they are not coprime?